Computing F

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We want to evaluate the function

$$F(a,b) = 2\sum_{k \geq 0} \frac{1}{(k+2)!(3+b/a)^{\overline{k}} a^k}$$

in the nice case where $b/a = m - 1/2$ for a nonnegative integer n. To this end, let

$$S(m,z) = \sum_{k \geq 0} \frac{(k+m+2)!}{(k+2)!(2k+2m+4)!} z^k.$$

From the arguments on page 11 of `https://arxiv.org/pdf/2004.00090.pdf`,

$$F(a,b) = \frac{2(4+2m)!}{(m+2)!} S(m, 4/a).$$

so we just need to evaluate $S(m,z)$.

Shift the summation index forward by 2:

$$S(m,z) = z^2 \sum_{k \geq 2} \frac{(k+m)!}{k!(2k+2m+2)!}$$

Note that $(k+m)!/k! = (k+m)^{\underline{m}} = (k+m)(k+m-1)\cdots(k+1)$, so

$$S(m,z) = z^{-2} \sum_{k \geq 2} \frac{(k+m)^{\underline{m}}}{(2k+2m+2)!} z^k.$$

If we now shift the summation index forward by m, we get

$$S(m,z) = z^{-2} z^{-m} \sum_{k \geq m+2} \frac{k^{\underline{m}}}{(2k)!} z^k.$$

Now we're in luck, because we "know"

$$\sum_{k \geq m+2} \frac{1}{(2k)!} z^k;$$

it is $\cosh \sqrt{z}$ minus finitely many terms.

It is easy to check that, for any polynomial $P(k)$,

$$\sum_{k\geq 0} P(k) a_k z^k = P(zD) \sum_{k\geq 0} a_k z^k,$$

where D is the differentiation operator. Thus,

$$\begin{aligned} S(m,z) &= z^{-m-2}(zD)^{\underline{m}} \sum_{k\geq m+2} \frac{z^k}{(2k)!} \\ &= z^{-m-2}(zD)^{\underline{m}}(\cosh\sqrt{z} - \sum_{0\leq k\leq m+1} \frac{z^k}{(2k)!}). \end{aligned}$$

This is about the best that we should hope for as a closed-form. But for fixed m, we can ask a computer to do some evaluations for us. For example,

$$S(2,z) = \frac{1}{z^2}\left(-\frac{1}{4}\sinh\left(\sqrt{z}\right) z^{-\frac{3}{2}} + \frac{1}{4\,z}\cosh\left(\sqrt{z}\right) - \frac{1}{12} - \frac{z}{120}\right)$$

If we take $a = 4$ with $m = 2$, then $b = 6$, and we get

$$\begin{aligned} F(4,6) &= \frac{2\cdot 8!}{4!} S(2, 4/4) \\ &= 3360((\cosh 1 - \sinh 1)/4 - 11/120) \\ &= \frac{840}{e} - 308. \end{aligned}$$

Maple programs to do this exist here: `https://rwdb.xyz/files/programs/computeF`