The Meta-C-Finite Ansatz

Robert Dougherty-Bliss

June 21, 2022

Abstract

The Fibonacci numbers satisfy the famous recurrence $F_n = F_{n-1} + F_{n-2}$. The theory of C-finite sequences ensures that the Fibonacci numbers whose indices are divisible by m, namely F_{mn}, satisfy a similar recurrence for every positive integer m, and these recurrences have an explicit, uniform representation. We will show that $a(mn)$ has a uniform recurrence over m for any C-finite sequence $a(n)$ and use this to automatically derive some famous summation identities.

The Fibonacci numbers F_n satisfy the famous recurrence $F_n = F_{n-1}+F_{n-2}$. The sequence which takes every *other* Fibonacci number, F_{2n}, satisfies the similar recurrence $F_{2n} = 3F_{2(n-2)} - F_{2(n-2)}$. In fact, every sequence of the form F_{mn} satisfies such a recurrence. Here are the first few:

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
F_{2n} &= 3F_{2(n-1)} - F_{2(n-2)} \\
F_{3n} &= 4F_{3(n-1)} + F_{3(n-2)} \\
F_{4n} &= 7F_{4(n-1)} - F_{4(n-2)} \\
F_{5n} &= 11F_{5(n-1)} + F_{5(n-2)}.
\end{aligned} \tag{1}
$$

If we look closely at the coefficients that appear—or plug them into the OEIS [7]—there seems to be a general recurrence:

$$F_{mn} = L_m F_{m(n-1)} + (-1)^{m+1} F_{m(n-2)}. \tag{2}$$

This conjecture is right on the money, and we can prove it a dozen different ways—Binet's formula, induction, generatingfunctionology—but the *outline* is more interesting.

We began with a sequence which satisfied a nice recurrence (F_n), examined recurrences for a family of related sequences (F_{mn}), then noticed that the coefficients on the recurrences satisfied a *meta pattern* (equation (2)). This outline holds for any sequence which satisfies a linear recurrence relation with constant coefficients. Such sequences are called *C-finite* [8, 4].

The remainder of the paper is organized as follows. Section 1 gives a brief overview of C-finite sequences, Section 2 proves that an analogue of (2) holds for any C-finite sequence, Section 4 shows that a similar property holds for products of C-finite sequences, and Section 3 applies some of our results to produce infinite families of summation identities.

1 The C-finite ansatz

The theory of C-finite sequences is beautifully laid out in [4] and [8]. What follows is a brief description of the principle results. For simplicity, assume that everything we do is over an algebraically closed field such as the complex numbers.

Given a sequence $a(n)$, let N be the shift operator defined by

$$Na(n) = a(n+1).$$

We say that $a(n)$ is *C-finite* if and only if there exists a polynomial $p(x)$ such that $p(N)a(n) = 0$ for all $n \geq 0$. We say that $p(x)$ *annihilates* $a(n)$. For example, $x^2 - x - 1$ annihilates the Fibonacci sequence $F(n)$ and $x - 2$ annihilates the exponential sequence 2^n. The set of all polynomials which annihilate a fixed $a(n)$ is an ideal. The generator of this ideal is the *characteristic polynomial* of $a(n)$, and we call its degree the degree (or order) of $a(n)$.

Every C-finite sequence has a closed-form expression as a sum of polynomials times exponential sequences. More specifically,

$$a(n) = \sum_{k=1}^{m} f_k(n) r_k^n,$$

where $r_1, r_2, \ldots, r_m$ are the distinct roots of the characteristic equation of $a(n)$ and $f_k(n)$ is a polynomial in n with degree less than or equal to the multiplicity of the root r_k. We call these formulas *Binet-type formulas* after Binet's famous formula for the Fibonacci numbers. For example, $(x-2)^2$ is an annihilating polynomial of any sequence $a(n)$ which satisfies the recurrence $a(n+2) = 4a(n+1) - 4a(n)$, and this implies $a(n) = (\alpha + \beta n)2^n$ for some constants α and β.

We can go the other way and derive an annihilating polynomial from a closed form expression. A term of the form $n^d r^n$ is annihilated by $(x-r)^{d+1}$, so for each exponential r^n in the closed form, look for the highest power n^d which is multiplied by r^n and write down $(x-r)^{d+1}$. For example, the sequence $a(n) = n3^n - \frac{n^2}{2} + 5^n$ is annihilated by $(x-3)^2(x-1)^3(x-5)$.

Finally, if $a(n)$ and $b(n)$ are two C-finite sequences, then so are the following:

$$a(n)b(n) \qquad a(n) \pm b(n) \qquad \sum_{k=0}^{n} a(k)b(n-k).$$

C-finite sequences are a special subclass of *holonomic sequences*, sequences which satisfy a linear recurrence with *polynomial coefficients* [3]. Holonomic sequences satisfy very similar properties, but do not have the readily computable closed forms which we need here.

2 Uniform recurrences

First up, we will prove the analogue of (2) for arbitrary C-finite sequences.

Proposition 1 If $a(n)$ is a C-finite sequence of order d, then $n \mapsto a(nm)$ satisfies a recurrence of the form

(3)
$$a(nm) = \sum_{k=1}^{d} c_k(m)a((n-k)m),$$

where $c_k(m)$ is C-finite with respect to m and has order at most $\binom{d}{k}$. The sequence $c_1(m)$ always satisfies the same recurrence as $a(n)$ itself, and $c_d(k) = \omega^k$, where ω is $(-1)^d$ times the constant coefficient of the characteristic polynomial of $a(n)$.

The following proof is constructive given the roots of the characteristic polynomial of $a(n)$, but [1] gives formulas for $c_k(m)$ in terms of partial Bell polynomials without reference to the roots.

Proof The Binet-type formula for $a(n)$ is a linear combination of terms of the form $n^i r^n$ where i is a nonnegative integer and r is a root of the characteristic polynomial of $a(n)$. Thus, the Binet-type formula for $a(nm)$ is a linear combination of terms of the form $(nm)^i r^{nm}$, which is equivalently a linear combination of terms of the form $n^i (r^m)^n$. The only thing that has changed is the exponential terms themselves, so if

$$\prod_{k=1}^{d} (x - r_k)$$

is the characteristic polynomial of $a(n)$ with possibly repeated roots $r_1, \ldots, r_d$, then

(4) $$\prod_{k=1}^{d}(x - r_k^m).$$

annihilates $n \mapsto a(nm)$. From the elementary theory of polynomials, the coefficients of (4) are elementary symmetric functions of the roots r_k^m. C-finite sequences are closed under multiplication and addition, so the coefficients of the polynomial are C-finite with respect to m.

To obtain the degree bound, recall that the coefficient on x^{d-i} in (4) equals $(-1)^i e_i(r_1^m, \ldots, r_d^m)$, where $e_i(r_1^m, \ldots, r_d^m)$ is the sum of all products of i distinct r_k^m. Each of these products is of the form α^m for some constant α. The number of such terms is an upper bound on the degree of the sequence with respect to m, and there are exactly $\binom{d}{i}$ of them.

Finally, note that the coefficient on x^{d-1} is precisely the sum $\sum_k r_k^m$, which is annihilated by the characteristic polynomial of $a(n)$ itself, and the coefficient on x^{d-d} is precisely the product $(r_1 r_2 \ldots r_d)^m$. ■

Example: Perrin numbers The Perrin numbers $P(n)$ are a third-order C-finite sequence defined by

$$P(0) = 0 \quad P(1) = 0 \quad P(2) = 2$$
$$P(n+3) = P(n+1) + P(n).$$

They are sometimes called the "skipponaci" numbers. They satisfy the interesting property that p divides $P(p)$ for every prime p. Tracing through the above proof reveals the meta-recurrence

(5) $$P(mn) = P(m)P(m(n-1)) + c(m)P(m(n-2)) + P(m(n-3)),$$

where $c(m)$ is A078712 in the OEIS.

Example: General second-order Let $a(n)$ be annihilated by $(x-r_1)(x-r_2)$ for distinct reals r_1 and r_2. The proof of Proposition 1 shows that $n \mapsto a(mn)$ is annihilated by

$$(x - r_1^m)(x - r_2^m) = x^2 - (r_1^m + r_2^m)x + (r_1 r_2)^m.$$

In particular, if r_1 and r_2 are the golden ratio and its conjugate, respectively, then $r_1^m + r_2^m = L_m$ is the mth Lucas number, and $r_1 r_2 = -1$. This recovers (1).

Example: Square Fibonacci The square Fibonacci numbers F_n^2 are also C-finite. Going through the steps of the above proof and consulting the OEIS reveals the following general identity:

(6) $$F_{mn}^2 = (5F_m^2 + 3(-1)^m)(F_{m(n-1)}^2 - (-1)^m F_{m(n-2)}^2) + (-1)^m F_{m(n-3)}^2.$$

Example: Tribonacci Consider the sequence T_n defined by

$$\begin{aligned} T_0 = 0 \quad T_1 = 0 \quad T_2 = 1 \\ T_n = T_{n-1} + T_{n-2} + T_{n-3}. \end{aligned}$$

The family of sequences $n \mapsto T_{nm}$ satisfy the following recurrences:

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ T_{2n} &= 3T_{2(n-1)} + T_{2(n-2)} + T_{2(n-3)} \\ T_{3n} &= 7T_{3(n-1)} - 5T_{3(n-2)} + T_{3(n-3)} \\ T_{4n} &= 11T_{4(n-1)} + 5T_{4(n-2)} + T_{4(n-3)} \\ T_{5n} &= 21T_{5(n-1)} + T_{5(n-2)} + T_{5(n-3)} \\ T_{6n} &= 39T_{6(n-1)} - 11T_{6(n-2)} + T_{6(n-3)}. \end{aligned}$$

In general,

$$T_{nm} = c_1(m)T_{(n-1)m} + c_2(m)T_{(n-1)m} + T_{(n-2)m},$$

where

$$\begin{aligned} c_1(1) = 1 \quad c_1(2) = 3 \quad c_1(3) = 7 \\ c_1(m) = c_1(m-1) + c_1(m-2) + c_1(m-3) \end{aligned}$$

and

$$\begin{aligned} c_2(1) = 1 \quad c_1(2) = 1 \quad c_1(3) = -5 \\ c_2(m) = -c_2(m-1) - c_2(m-2) + c_2(m-3). \end{aligned}$$

The sequences $c_k(m)$ were found via guessing. However, Proposition 1 establishes that these sequences *are* C-finite, and so proving our guess requires that we check only finitely many terms. In this case we must check no more than double the maximum degree, which is 6 terms. We have produced just enough examples above to constitute a proof.

3 Uniform sums

The Fibonacci numbers satisfy the famous summation identity

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1. \tag{7}$$

There are as many ways to prove this identity as there are articles devoted to evaluating related Fibonacci sums [5, 6, 2], but the most useful method at this juncture is the following method outlined in [4]. The annihilating polynomial of F_n can be written as

$$x^2 - x - 1 = (x-1)x - 1.$$

Applying this to F_n shows that $F_n = (x-1)F_{n+1} = F_{n+2} - F_{n+1}$. If we sum over n, then the right-hand side telescopes and we recover (7). In general, if $p(x)$ annihilates $a(n)$ and $p(1) \neq 0$, then we can write $p(x) = (x-1)q(x) + p(1)$ for some easily-computable polynomial $q(x)$. Applying this to $a(n)$ shows that $a(n) = (x-1)b(n)$ where $b(n) = -q(x)a(n)/p(1)$. Summing over n yields

$$\sum_{0 \leq k < n} a(k) = b(n) - b(0).$$

From this idea, the uniform recurrences we have derived for sequences of the form $n \mapsto a(mn)$ and $n \mapsto a(ni)a(nj)$ will help us discover uniform summation identities.

Here is one such identity for the Perrin numbers, using (5).

Proposition 2 The Perrin numbers $P(n)$ satisfy

$$\sum_{0 \leq k < n} P(mn) = \frac{(P(n)-3)(1-P(m)-c(m)) + P(n+1)(1-P(m)) + P(n+2) - 2}{P(m)+c(m)},$$

where $c(m)$ is A078712 in the OEIS.

Using (6), we can quickly rediscover the following infinite family of sums for the square of the Fibonacci numbers.

Proposition 3 If m is odd, then

$$\sum_{0 \leq k < n} F_{mk}^2 = \frac{F_{mn}F_{m(n-1)}}{L_m}.$$

Proof Using (6), we obtain

$$\sum_{0\le k<n} F_{mk}^2 = \frac{F_{mn}^2(7-10F_m^2) + (F_{m(n+1)}^2 - F_m^2)(4-5F_m^2) + F_{m(n+2)}^2 - F_{2m}^2}{10F_m^2 - 8}.$$

This is far from the most economical representation. First, the numerator here contains $(5F_m^2-4)F_m^2 - F_{2m}^2$. It is easy to check that

$$(5F_m^2-4)F_m^2 - F_{2m}^2 = -8F_m^2\frac{(-1)^m+1}{2}, \tag{8}$$

so the expression on the left vanishes when m is odd. We are down to

$$\frac{F_{mn}^2(7-10F_m^2) + F_{m(n+1)}^2(4-5F_m^2) + F_{m(n+2)}^2}{10F_m^2 - 8}.$$

Applying the general recurrence (1) to $F_{m(n+2)}$ and simplifying the result brings us to

$$\frac{F_{mn}^2(8-10F_m^2) + F_{m(n+1)}^2((4-5F_m^2)+L_m^2) + 2F_{mn}L_mF_{m(n+1)}}{10F_m^2 - 8}.$$

When m is odd, the identity $4-5F_m^2+L_m^2 = 0$ follows from dividing (8) by F_m^2 and recalling that $L_m = F_{2m}/F_m$. Using this and simplifying gives

$$\frac{F_{mn}(-L_mF_{mn} + F_{m(n+1)})}{L_m},$$

and applying the general recurrence (1) once more to $F_{m(n+1)}$ gives us the final answer $F_{mn}F_{m(n-1)}/L_m$. ■

4 Uniform products

The proof of Proposition 1 relied on little more than the identity $r^{mn} = (r^m)^n$ and some structural facts about C-finite sequences. Unsurprisingly, these ideas apply to other settings. The below proposition shows how to apply the idea to prove that sequences of the form $n \mapsto a(ni)a(nj)$ also satisfy meta C-finite recurrences.

Proposition 4 If $a(n)$ is C-finite of degree d whose characteristic polynomial has m distinct roots, then $P_{i,j}(n) = a(ni)a(nj)$ satisfies a recurrence of the form

$$P_{i,j}(n) = \sum_{k=1}^{m(2d-m)} c_k(i,j)P_{i,j}(n-k),$$

where each $c_k(i,j)$ is C-finite with respect to i and j and $c_k(i,j) = c_k(j,i)$. The sequence $c_k(i,j)$ has order (with respect to i or j) no more than $\binom{d}{k}$.

Proof Write the characteristic polynomial of $a(n)$ as $\prod_{k=1}^{m}(x-r_k)^{d_k+1}$ where the r_k are distinct and $d_1 + d_2 + \cdots + d_m = d - m$. Then,

$$a(n) = \sum_{k=1}^{m} p_k(n) r_k^n,$$

where p_k is a polynomial in n of degree d_k or less. Therefore

$$P_{i,j}(n) = \sum_{1 \le k,v \le m} p_k(in) p_v(jn) (r_k^i r_v^j)^n.$$

Immediately, we see that $P_{i,j}(n)$ is annihilated by

$$\prod_{1 \le k,v \le m} (x - r_k^i r_v^j)^{d_k + d_v + 1}, \tag{9}$$

a polynomial of degree $\sum_{k,v}(d_k + d_v + 1) = m(2d - m)$. The coefficients of this polynomial are elementary symmetric polynomials in the variables $\{r_k^i r_v^j\}_{1 \le k,v \le d}$, and therefore C-finite with respect to i and j by the C-finite closure properties. The roots $r_k^i r_v^j$ are symmetric in i and j, so the coefficient sequences are as well.

The coefficient on x^{D-k} is essentially the sum of all products of k distinct elements from $\{r_k^i r_v^j\}_{1 \le k,v \le d}$. As a sequence in i the r_v^j factors are irrelevant: The coefficient will be annihilated by the characteristic polynomial for the sum of all products of k distinct elements from $\{r_k^i\}_{1 \le k \le d}$. Each term of this latter sum is of the form α^i for some constant α, and there are no more than $\binom{d}{k}$ distinct values of α. Therefore $c_k(i,j)$ has order no more than $\binom{d}{k}$ with respect to i (and also j). ■

The previous proof can be slightly modified to produce a stronger statement. Namely, if we split the product (9) into diagonal and off-diagonal terms, we get the following corollary.

Corollary 1 *Let $a(n)$ be a C-finite sequence of degree d whose characteristic polynomial has m distinct roots. Then $n \mapsto a(ni)a(nj)$ is annihilated by a polynomial $C_{i,j}(x)$ which factors as*

$$C_{i,j}(x) = L_{i+j}(x) R_{i,j}(x), \tag{10}$$

where $\deg L_{i+j} = 2d - m$ and $\deg R_{i,j} = (m-1)(2d-m)$. The coefficients of $L_{i+j}(x)$ are C-finite sequences in $i+j$ and the coefficients of $R_{i,j}(x)$ are C-finite sequences which are symmetric in i and j.

There is one case of this corollary worth highlighting. Now that we know these annihilating polynomials with C-finite coefficients exist, we could find them by computing enough examples and guessing a pattern. However, if the degrees of $L_{i+j}(x)$ and $R_{i,j}(x)$ are the same, then it is not always clear which factor is L and which factor is R in a given example. This happens when $2d-m = (m-1)(2d-m)$. Since $m \leq d$, the interesting solution is $m = 2$. Thus sequences with exactly two roots in their characteristic polynomial should be handled "manually." We will show one example.

Example: Second-order annihilators Let $a(n)$ be a C-finite sequence annihilated by the quadratic $(x - r_1)(x - r_2)$ where $r_1 \neq r_2$. Then $n \mapsto a(ni)a(nj)$ is annihilated by

$$(x^2 - \mathcal{L}(i + j)x + (r_1 r_2)^{i+j})(x^2 - (r_1 r_2)^j \mathcal{L}(i - j)x + (r_1 r_2)^{i+j})$$

where $\mathcal{L}(n) = r_1^n + r_2^n$. If $a(n) = F(n)$ equals the nth Fibonacci number, then $\mathcal{L}(n) = L(n)$ is the nth Lucas number, $r_1 r_2 = -1$, and we obtain the annihilator

$$(x^2 - L(i + j)x + (-1)^{i+j})(x^2 - (-1)^j L(i - j)x + (-1)^{i+j}).$$

5 Computer demo

This article is joined by a corresponding Maple package `MetaCfinite`. With `MetaCfinite`, nearly all the propositions described in this article can be explored and checked empirically.

Guessing uniform recurrences Suppose that we want to discover (1) and the corresponding general pattern. The following Maple commands compute the five recurrences from (1):

```
Fib := [[0, 1], [1, 1]:
mSect(Fib, 1, 0); # [[0, 1], [1, 1]]
mSect(Fib, 2, 0); # [[0, 1], [3, -1]]
mSect(Fib, 3, 0); # [[0, 2], [4, 1]]
mSect(Fib, 4, 0); # [[0, 3], [7, -1]]
mSect(Fib, 5, 0); # [[0, 5], [11, 1]]
```

We are trying to guess the pattern followed by $1, 3, 4, 7, 11$, and $1, -1, 1, -1, 1$. The following command does this for us:

```
MetaMSect(Fib, 0); # [[[1, 3], [1, 1]], [[1], [-1]]]
```

This tells us that, for example, the coefficient on $F_{m(n-1)}$ is a sequence L_m which begins $L_1 = 1$, $L_2 = 3$, and satisfies $L_m = L_{m-1} + L_{m-2}$. These are the Lucas numbers.

Uniform summation identities The procedure `polysum(a, n, p, x` computes an expression for $\sum_{0 \leq k < n} a(k)$ where $a(n)$ is a C-finite sequence with characteristic polynomial $p(x)$. For example, the following command derives the famous identity (7):

```
polysum(F, n, x^2 - x - 1, x); # F(n + 1) - F(1).
```

This is most powerful when joined with uniform recurrences found by `MetaMSect`. For instance, the sequence $n \mapsto F(mn)$ has characteristic polynomial $p_m(x) = x^2 - L(m)x - (-1)^{m+1}$. The following commands derive a summation identity for $\sum_{0 \leq k < n} F(mk)$:

```
polysum(Fm, n, x^2 - L(m) * x - (-1)^(m + 1), x);
      (Fm(n) - Fm(0)) (1 - L(m)) + Fm(n + 1) - Fm(1)
    - ----------------------------------------------
                                        m
                          1 - L(m) + (-1)
```

That is, we have automatically derived the famous identity

$$\sum_{0 \leq k < n} F(mk) = \frac{F(mn)(1 - L(m)) + F(m(n+1)) - F(m)}{L(m) - 1 - (-1)^m}.$$

6 Conclusion

We have used the theory of C-finite sequences to establish *meta-facts* about the recurrences C-finite sequences satisfy. Namely, we have shown that the recurrences satisfied by $n \mapsto a(nm)$ and $n \mapsto a(ni)a(nj)$ are uniform in a C-finite sense. This allowed us to state uniform families of summation identities for some C-finite sequences.

The summation identities our methods derive are automatic and uniform, but we do not claim that they are the "best possible." For instance, the first expression obtained for $\sum_{k=0}^{n-1} F_{mk}^2$ in Proposition 3 is quite cumbersome compared to the final answer:

$$\frac{F_{mn}^2(7 - 10F_m^2) + (F_{m(n+1)}^2 - F_m^2)(4 - 5F_m^2) + F_{m(n+2)}^2 - F_{2m}^2}{10F_m^2 - 8} = \frac{F_{mn}F_{m(n-1)}}{L_m}.$$

It still takes some (semi-automatic) sweat to discover this reduction. Can we automatically discover and prove such "complex = simple" identities? And might this apply to more complex sums, such as $\sum_{k=0}^{n-1} F_{mk}^5$? The answer is likely yes—and perhaps a C-finite simplification algorithm already exists—but we leave this as an open problem.

Finally, the author would like to acknowledge Doron Zeilberger for bringing these problems to his attention and providing encouragement.

References

[1] Birmajer, D., Gil, J.B. and Weiner, M.D., 2015. Linear recurrence sequences with indices in arithmetic progression and their sums. arXiv preprint arXiv:1505.06339.

[2] Frontczak, R., 2018. Sums of powers of Fibonacci and Lucas numbers: A new bottom-up approach. Notes on Number Theory and Discrete Mathematics, 24(2), pp.94-103.

[3] Kauers, Manuel, 2013. The Holonomic Toolkit. In Computer Algebra in Quantum Field Theory (pp. 119-144). Springer, Vienna.

[4] Kauers, Manuel and Paule, Paule. (2010). *The Concrete Tetrahedron: Symbolic sums, Recurrence Equations, Generating Functions, Asymptotic Estimates.* Springer.

[5] Layman, J.W., 1977. Certain general binomial-Fibonacci sums. Fibonacci Quart, 15(3), pp.362-366.

[6] Melham, R., 1999. Sums involving Fibonacci and Pell numbers. Portugaliae Mathematica, 56(3), pp.309-318.

[7] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, Published electronically at `http://oeis.org`.

[8] Zeilberger, Doron., 2013. The C-finite ansatz. The Ramanujan Journal, 31(1), pp. 23-32.