Workshop Notes

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31 March 2021

Let's start off with something everyone wants to hear about: the midterm!

Midterm grading has begun. We hope to be done by Friday. That being said, let me go ahead and mention some preliminary things I saw that worried me.

Exercise 1 Let $A = \{a_1, a_2, \dots\}$ be an arbitrary countable subset of $\mathbf{R}$. Define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$\begin{aligned} f(a_n) &= \frac{1}{n} \\ f(x) &= 0 \text{ if } x \notin A. \end{aligned}$$

Find D_f, the set of discontinuities of f.

Solution 1

Nearly everyone correctly said that $D_f = A$. However, almost no one actually proved this.

Here is, roughly, what most people did:

> Consider an arbitrary $a_n \in A$. Since A is countable, every neighborhood of a_n contains points of A^C, and therefore $f(x) = 0$ for some x in every neighborhood of a_n. Since $f(a_n) = 1/n > 0$, it follows that f is discontinuous at a_n.

This proves that A is a *subset* of D_f, but not that A *equals* D_f. (*Note:* We don't need to say anything about limit points of A for this part.)

Here is the other part, that *very few* people did, but you should have done to get full credit:

Let $x \notin A$. Let x_n be a sequence in $\mathbf{R}$ which converges to A. For any positive integer M there exists a positive integer N such that, if $n > N$, then $x_n = a_m$ only if $m > M$. (Why? There are only finitely many a_m with $m \leq M$, so the set $\{n \mid x_n = a_m, m \leq M\}$ is finite. Let N be the maximum of this set.) Therefore, for $n > N$, either $x_n \notin A$ (and $f(x_n) = 0$) or $x_n = a_m$ and $f(x_n) = 1/m < 1/M$. In either case, $f(x_n) < 1/M$ for $n > N$, which shows that $\lim_n f(x_n) = 0$.

Exercise 2 Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution 2
The real crux of this problem is showing that one-sided limits exist at every point for a monotone function. I'll prove half of this.

Theorem 1. *If $f\colon \mathbf{R} \to \mathbf{R}$ is monotonically increasing, then $\lim_{x\to c^-} f(x)$ exists for every $c \in \mathbf{R}$.*

Proof. Let $A = \{f(x) \mid x < c\}$. We have $f(x) \leq f(c)$ for all $x < c$, therefore A is bounded from above. I claim that

$$\lim_{x\to c^-} f(x) = \sup A.$$

Indeed, for each $\epsilon > 0$, there exists some $f(x^*) \in A$ such that

$$\sup A - \epsilon < f(x^*) \leq \sup A$$

Then $x \in (x^*, c)$ implies

$$\sup A - \epsilon < f(x^*) \leq f(x) \leq \sup A$$

so $|f(x) - \sup A| < \epsilon$ for all $x \in (x^*, c)$. Thus $\lim_{x\to c^-} f(x) = \sup A$ with $\delta = c - x^*$. □

Right-hand limits exist by an analogous argument with the infimum rather than the supremum. Monotonically decreasing functions work exactly the same.

Now that one-sided limits exist, you have to check that removable discontinuities are impossible, and that leaves only jump discontinuities.

Two other things I wanted to say today.

1. Let f be a continuous function on A. If there exists a compact set K which contains A, and f has a *continuous extension* to K, then f is uniformly continuous on A.

   What exactly does this mean?

   A function $g\colon K \to \mathbf{R}$ is a *continuous extension* of $f\colon A \to \mathbf{R}$ if $A \subseteq K$, the function g is continuous on K, and $g(x) = f(x)$ for all $x \in A$.

   Once you have this extension, the proof is easy! It's just an application of the "continuous on compact implies uniformly continuous" theorem. Actually doing the extending is not so easy.

2. I really rushed through my last example last week. Let me try it again.

   Let y_0 be your favorite real number. Then, let

$$\begin{aligned} y_1 &= \sin(y_0/2) \\ y_2 &= \sin(y_1/2) \\ y_3 &= \sin(y_2/2) \\ y_4 &= \sin(y_3/2), \end{aligned}$$

   and so on. In general,
$$y_{n+1} = \sin(y_n/2)$$
   for $n \geq 0$. For argument's sake, let's take $y_0 = 1$. Then

$$\begin{aligned} y_1 &= \sin 1/2 \approx 0.4794 \\ y_2 &= \sin(\sin(1/2)/2) \approx 0.2374 \\ y_3 &\approx 0.1184 \\ y_4 &\approx 0.0592. \end{aligned}$$

   These numbers are getting closer and closer to 0!

   Let's pretend for a moment that y_n has a limit, say $\lim_n y_n = y$. Since $\sin(x/2)$ is continuous, we can let $n \to \infty$ in both sides of

$$y_{n+1} = \sin(y_n/2)$$

   to obtain
$$y = \sin(y/2).$$

That is, y is a *fixed point* of $f(x) = \sin(x/2)$, meaning that $f(y) = y$. It is not hard to see that $y = 0$ is the only fixed point of $\sin(x/2)$, so if y_n converges, it must converge to 0. Our empirical evidence suggests that it *does* converge, but this is not a proof.

The *proof* of this fact is an application of the *contraction mapping theorem.*

Theorem 2. *Let f be a function such that, for some $c \in (0, 1)$,*

$$|f(x) - f(y)| \leq c|x - y|$$

for all real x and y. Then there exists a unique *real t such that $t = f(t)$. Further, for any real t_0, the sequence defined by*

$$t_{n+1} = f(t_n)$$

converges to t.

The proof of this is an exercise in Abbott. It relies on the following key lemma.

Lemma 1. *If $|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$ for $c \in (0, 1)$, then a_n is Cauchy.*

Proof. Exercise! □