Workshop Notes

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Logistics

Nothing to say here.

New stuff

This week you learned about about the real numbers. You already knew most of the algebraic structure of the reals (addition, multiplication, inequalities, etc.), but the real kicker is the *completeness axiom*.

Consider the set $(0, 1)$ of all reals between 0 and 1. Note that $x \leq 2$ for every $x \in (0, 1)$. That is, 2 is an *upper bound* for $(0, 1)$. There are many such upper bounds: $1 + 1/2$, and $1 + 1/3$, and $1 + 1/999999999$, and so on. The completeness axiom tells us that there is a *least* one, i.e., an upper bound which we cannot "improve" by making it smaller. We call this least upper bound the *supremum* of $(0, 1)$, and denote it $\sup\,(0, 1)$. Here, $\sup\,(0, 1) = 1$.

The axiom of completeness is silly for $(0, 1)$—obviously the least upper bound exists in this case—but it is *crucial* in cases where the least upper bound is not obvious. I don't want to spoil Professor Coley's thunder, so I'll just say that the completeness axiom is the most important property of the reals.

The axiom is also a good philosophical example of *infinity*. If there were only finitely many upper bounds of $(0, 1)$, then we could go through the list and pick the smallest. Since there are infinitely many we can't do this "by hand," but the axiom says that we can do it abstractly. Trying to pick the smallest element of an infinite set is a recurring problem throughout analysis.

Now I'm going to open the floor to questions and discussion from everyone. If discussion flows, perfect. Otherwise, I selected a few exercises.

Exercise 1

(a) What is $\sup \mathbf{R}$?

(b) Can you construct a set S with $\inf S \geq \sup S$? If yes, what are *all* such sets? If no, prove that it is impossible.

(c) Can you construct a nonempty set, bounded from above, that does not contain its supremum?

(d) Can you construct a nonempty set of *integers*, bounded from above, that does not contain its supremum?

(e) Can you construct a nonempty set bounded from above where the supremum is not obvious?

Exercise 2 (Abbott, 1.4.8)

The nested interval theorem says that $\bigcap_{k\geq 1} I_k$ is nonempty if $\{I_k\}$ is a sequence of nonempty, bounded, nested, closed intervals. Do we need all of those conditions? That is, what happens if the intervals are open? Or not nested? Or unbounded? The following questions explore some of these ideas.

Give an example of each of the following, or argue that it is impossible.

(a) *Disjoint*, nonempty, bounded from above sets A and B such that $\sup A = \sup B$, $\sup A \notin A$, and $\sup B \notin B$.

(b) A sequence of nested, *open* intervals $I_1 \supseteq I_2 \subseteq \cdots$ with $\bigcap_{k\geq 1} I_k$ having a nonzero but *finite* number of elements.

(c) A sequence of nested, *unbounded*, closed intervals $L_1 \supseteq L_2 \supseteq \cdots$ with $\bigcap_{k\geq 1} L_k$ empty.

(d) A sequence of closed, bounded intervals $J_1, J_2, \ldots$ such that $\bigcap_{k=1}^{n} J_k$ is nonempty for every positive integer n, yet $\bigcap_{k\geq 1} J_k$ is empty.

Exercise 3 (Abbott, 1.3.4.) [I don't expect that we'll have time to get to this, so I'm not going to bother typing it up.]