Workshop Notes

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Logistics

Nothing to say here.

New stuff

Since I last saw you, you learned about cardinality and sequences. Sequences will get properly confusing when we get to some theorems, but for now I imagine thatt cardinality is very weird for everyone. Let's remind ourselves of what's going on.

One way to measure a set is to just count all the things in it;

$$|\{a, b, c\}| = 3.$$

Easy. Another way is to "line up" one set with another. For example:

$$\begin{gathered}\{a, b, c\} \\ \{1, 2, 3\}\end{gathered}$$

These sets are the same size because we can line them up. If we agree that $\{1, 2, 3\}$ has size 3, then $\{a, b, c\}$ has size 3 as well.

This second way seems like a really stupid way to measure sets. Until you learn about infinite sets. You can't "count" infinite sets. It doesn't make any sense. You could say that every infinite set has size ∞, but you're going to miss a *weird* thing that happens: infinite sets can have different sizes if you measure using the second way.

Formally, we say that two sets A and B have the same cardinality if there exists a bijection between them (a way to line them up). That is, there's a function $f\colon A \to B$ such that:

1. For all $b \in B$, there is some $a \in A$ such that $f(a) = b$ (surjectivity); and

2. If $f(a_1) = f(a_2)$, then $a_1 = a_2$ (injectivity).

For example, $\{1, 2, 3, \dots\}$ has the same size as $\{2, 3, 4, \dots\}$, because we can define

$$f(n) = n + 1,$$

which is a bijection. (Exercise!)

More interestingly, $\{1, 2, 3, \dots\}$ has the same size as $\{2, 4, 6, 8, \dots\}$. (Define $f(n) = 2n$.) That is, the even natural numbers have the same size as all the natural numbers themselves! The odds *also* have the same size as the natural numbers. (Define $f(n) = 2n + 1$.) If we say that $\aleph_0$ is the "size" of $\{1, 2, 3, \dots\}$ then we have an equation like

$$\aleph_0 + \aleph_0 = \aleph_0.$$

That's weird. It's not so weird if we think of $\aleph_0$ as ∞, but there are *other* infinities! Famously, $\mathbf{R}$ is *uncountable*, in the sense that there is no bijection from $\mathbf{N}$ to $\mathbf{R}$. (See Abbott.) But clearly $\mathbf{R}$ is bigger than $\mathbf{N}$... this is all very weird.

I won't say anything more—that's what Abbott is for—but I'm sure that you have lots of questions about cardinality and homework. Fire away!

Promises

I forgot to prove something for you that I promised I would. To make it up to you, I'll show you something fancier to whet your appetite for things to come. (I won't talk about this at all during the workshop. This is just a bonus read if you're interested.)

Theorem 1. *If S is a nonempty, bounded set of integers, then* $\sup S$ *is contained in S.*

This is actually a special case of a more general result. First, I need to define a new object.

Definition 1. A point x is a *limit point* of a set S if every ϵ-neighborhood contains a point of S not equal to x.

For example, 0 is a limit point of $\{1, 1/2, 1/3, 1/4, \dots\}$.

A nice fact about sups and infs is that they're either in the underlying set or limit points of it.

Proposition 1. *If* $\sup S \notin S$*, then* $\sup S$ *is a limit point of* S*.*

Proof. For every $\epsilon > 0$, we can find some $s \in S$ such that $\sup S - \epsilon < s < \sup S$, the last inequality being strict since $\sup S \notin S$. Thus every ϵ-neighborhood of $\sup S$ contains a point of S not equal to $\sup S$, so $\sup S$ is a limit point of S. □

Corollary 1. *Let* S *be a nonempty set, bounded from above, without limit points. Then* $\sup S \in S$*.*

Proof. Since S has no limit points, we must have $\sup S \in S$ by the previous proposition. □

Now Theorem 1 is easy!

Proof of Theorem 1. If S is a set of integers, then S has no limit points. (Every element is at least 1 away from every other element.) Thus, if $\sup S$ exists, $\sup S \in S$ by the previous corollary. □

All of this is the application of general *metric space theory* to $\mathbf{R}$. It turns out that a lot of arguments we make about sequences, sets, and so on, generalize to other situations without any modifications. If this sounds interesting, go look up "metric spaces."