Workshop Notes

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Logistics

Nothing to say here.

New stuff

Since I last saw you, you learned more about sequences and got some new homework. We're going to talk about some sequence topics and a problem from the new homework, as well as any other questions you might have.

Cesàro means

If we have a sequence $x(n)$, then we can take *averages* of it by defining

$$y(n) = \frac{1}{n} \sum_{k=1}^{n} x(k).$$

Plausible statement: If $x(n)$ converges, then the averages also converge to the same thing.

This is true. It's called Cesàro summation. The proof illustrates one of the main techniques of analysis: Splitting a problem into two managable parts.

Theorem 1. *If $x(n) \to x$, then*

$$y(n) = \frac{1}{n} \sum_{k=1}^{n} x(k) \to x.$$

Proof. We want to show that, given any $\epsilon > 0$, there exists some integer N such that

$$|y(n) - x| < \epsilon$$

for $n > N$.

The trick about averaging is that we can break up a single x into many terms:

$$y(n) - x = \frac{1}{n}\sum_{k=1}^{n} x(k) - x = \frac{1}{n}\sum_{k=1}^{n}(x(k) - x).$$

Therefore, by the triangle inequality,

$$|y(n) - x| \leq \frac{1}{n}\sum_{k=1}^{n} |x(k) - x|.$$

For "large" k, we have the "sharp" bound $|x(k) - k| < \epsilon$. For "small" k, we only have the "coarse" fact that $|x(k) - k|$ is bounded, since $x(k) - k \to 0$.

To make this idea rigorous, pick M such that $|x(k) - x| \leq M$ for all k (by boundedness), and N such that $|x(k) - x| < \epsilon$ for $k > N$ (by convergence). Then, for $n > N$,

$$\begin{aligned} |y(n) - x| &\leq \frac{1}{n}\sum_{k=1}^{N} |x(k) - x| + \frac{1}{n}\sum_{N<k\leq n} |x(k) - x| \\ &\leq \frac{NM}{n} + \frac{\epsilon(n - N)}{n} \\ &\leq \frac{NM}{n} + \epsilon. \end{aligned}$$

To get the right bound here, we need to turn that NM/n into something involving ϵ. But N and M are both fixed relative to ϵ, so we can just require n to be really big in comparison. That is, let N' be such that $N' > N$ and $NM/n < \epsilon$ for $n > N'$. Then, for $n > N'$, the bound becomes

$$|y(n) - x| \leq \epsilon + \epsilon = 2\epsilon.$$

Since ϵ was arbitrary, this is equivalent and shows that $y(n) \to x$. □

Interesting question: If the averages converge, does $x(n)$ converge?

Limits superior and inferior

Not every sequence has a limit, but sometimes we want to talk about "limits" even when they don't exist. For example, $a(n) = (-1)^n$ does not "converge," but it kind of has two "pseudo-limits," 1 and -1. These "peusdo-limits" are made precise by the *limits superior and inferior*.

Given any sequence $a(n)$, there exists a number β (possibly ∞) such that:

1. For every $\epsilon > 0$, we have $a(n) < \beta + \epsilon$ eventually.

2. For every $\epsilon > 0$, there are infinitely many n such that $a(n)$ is in $(\beta - \epsilon, \beta]$.

The number β is called the *limit superior* of $a(n)$, and we denote it by $\beta = \limsup_n a(n)$.

The above properties actually uniquely define the limit superior, but they don't prove that it exists. Let's go through how Abbott does it. For now, suppose that $a(n)$ is a bounded sequence.

(a) Prove that $y(n) = \sup\{a(k) \mid k \geq n\}$ converges. Let $\limsup_n a(n) = \lim_n y(n)$.

(b) Give the analogous definition for $\liminf_n a(n)$.

(c) Show that $\liminf_n a(n) \leq \limsup_n a(n)$. [Hint: If $x(n)$ and $y(n)$ converge and $x(n) \leq y(n)$, then $\lim_n x(n) \leq \lim_n y(n)$. (You should prove this!)]

(d) Show that $\lim_n a(n) = x$ iff $\liminf_n a(n) = x = \limsup_n a(n)$. (You can't use the above properties I mentioned unless you prove them!)

Exercise 1 If $a(n) \leq b(n)$, we can't always say that $\lim_n a(n) \leq \lim_n b(n)$, because the limits might not even exist. Show that

$$\limsup_n a(n) \leq \limsup_n b(n)$$

and

$$\liminf_n a(n) \leq \liminf_n b(n)$$

whenever $a(n) \leq b(n)$ for all n.