Workshop Notes

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Our last workshop! It's been a strange semester. It feels like it's been unbearably long and frighteningly short at the same time. Oh well. We're here together for one last time.

Some notes:

1. Don't forget to take your SIRS surveys! In fact, I'll set aside five starting now for everyone to do the surveys.

2. Last midterm due tomorrow. Hopefully you started before today!

3. If I held a review session during class next week, would anyone want to come? (Not mandatory at all.)

There was a homework problem that asked you about these propositions:

(a) If $f_n \to f$ uniformly and each f_n has at most M discontinuities, then f has at most M discontinuities.

(b) If $f_n \to f$ uniformly and each f_n has at most countably many discontinuities, then f has at most countably many discontinuities.

These are both true! But they're a little tricky.

For (a): Suppose that each f_n is discontinuous at exactly *one* point, and they all have the same discontinuity. If we just tossed that point out, then the "new" f_n would converge uniformly and be continuous, so the limit function is continuous.

The problem is that we have no guarantees that the discontinuities all happen at the same point. What if they "move around" and sneak in that way? Here's an important fact that says that that can't happen.

Theorem 1. *If $f_n \to f$ uniformly and f is discontinuous at x, then f_n is discontinuous at x for sufficiently large n.*

In words, the only way for a uniform limit to be discontinuous is for the sequence functions to eventually be discontinuous at the same point.

Proof. By assumption, there exists an $\epsilon > 0$ and a sequence $x_m \to x$ such that $|f(x) - f(x_m)| \geq \epsilon$ for all m. Choose an N such that $|f_n(t) - f(t)| < \epsilon/3$ if $n > N$ for all t. Then, by the triangle inequality,

$$\begin{aligned}\epsilon &\leq |f(x) - f(x_m)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_m)| + |f_n(x_m) + f(x_m)| \\ &< \epsilon/3 + |f_n(x) - f_n(x_m)| + \epsilon/3.\end{aligned}$$

In other words,

$$|f_n(x) - f_n(x_m)| + 2\epsilon/3 \geq \epsilon,$$

so

$$|f_n(x) - f_n(x_m)| \geq \epsilon/3$$

for all m, if $n > N$. This says that f_n is discontinuous at x for $n > N$. $\square$

For (b), we need to put a crude upper bound on the number of discontinuities.

Theorem 2. *Let D_g be the set of discontinuities of a function g. If $f_n \to f$ uniformly, then $D_f \subseteq \bigcup_{n\geq 1} D_n$.*

Proof. Taking complements, Our claim is equivalent to

$$\bigcap_{n\geq 1} D_n^C \subseteq D_f^C.$$

This says, "if every f_n is continuous at x, then f is continuous at x," which is true when $f_n \to f$ uniformly. $\square$

Countable unions are countable!

1 New stuff

Does the following work?

$$\lim \frac{d}{dx} = \frac{d}{dx} \lim$$

Well, sometimes.

Example 1

$$g_n(x) = \frac{nx + x^2}{2n}.$$

With this function, we have

$$\lim_n \frac{d}{dx} \frac{nx + x^2}{2n} = \lim_n \frac{n + 2x}{2n} = \frac{1}{2}$$

and

$$\frac{d}{dx} \lim_n \frac{nx + x^2}{2n} = \frac{d}{dx} \frac{x}{2} = \frac{1}{2}.$$

It works!

Example 2 Let

$$f_n(x) = \frac{x^n}{n}$$

on $[0, 1]$. Then,

$$\lim_n \frac{d}{dx} \frac{x^n}{n} = \lim_n x^{n-1} = [x = 1]$$

and

$$\frac{d}{dx} \lim_n \frac{x^n}{n} = \frac{d}{dx} 0 = 0.$$

Yikes!

Look at the derivative x^{n-1}. We know that these functions don't converge uniformly—they're our favorite examples. These kinds of problems are what can happen when you ignore uniform convergence of the derivatives.

Example 3

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

1. $h_n \to 0$ uniformly (easy!)

2. $h_n'(x)$ diverges for every x, so it doesn't even make *sense* to take a limit!