Number Theory Homework I

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This homework is meant to be a warmup for things covered in 300 along with a peek at some number theory we covered in the first week. Direct proofs, inequalities, induction, the contrapositive, and perhaps most importantly, *style*.

You will get the most out of the homework if you try to solve the questions on your own before anything else. That being said, you are always free to discuss homework with your classmates or ask me about it. Just make sure your writeup reflects *your* understanding.

EXPECTATIONS Write your proofs with *good style*. What is good style? Concise yet deep. Simple yet elegant. Practical yet fun. Basically, go read the first chapter of Knuth's masterpiece and buy yourself a copy of Strunk and White's *The Elements of Style*. Your efforts will be repayed tenfold.

For example, suppose you were proving "the sum of even integers is even."

Proof [Terrible proof]

$$\begin{aligned}
&\exists n, k\ [2n + 2k] \\
&\implies 2n + 2k = 2(n + k) \\
&\exists m\ [n + k = m] \\
&\therefore 2n + 2k = 2m
\end{aligned}$$

■

Proof [Excellent proof] If x and y are even integers, then there exist integers n and k such that $x + y = 2n + 2k = 2(n + k)$. Since $n + k$ is an integer, $x + y$ is even. ■

Both proofs are "correct," but the first one makes me dizzy. Bring your readers joy, not pain.

Exercises

In general, assume that variables like n, m, and k are integers.

Exercise 1

- (**a**) Prove that n^2 is even if and only if n is even.
- (**b**) If $\sqrt{2} = a/b$ for coprime integers a and b, then $2b^2 = a^2$. Use the previous part to derive a contradiction about a and b, and conclude that $\sqrt{2}$ is irrational.

Solution 1

- (**a**) (Ungraded) If $n = 2k$, then $n^2 = 4k^2 = 2(2k^2)$. On the other hand, if $n = 2k + 1$, then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- (**b**) (Ungraded) Since $2b^2 = a^2$, we see that a^2 is even. The previous part implies that a is even, say $a = 2k$. Then $2b^2 = 4k^2$, and dividing by 2 yields $b^2 = 2k^2$. But then b is even, and that contradicts that a and b are coprime.

Exercise 2 Three natural numbers $x < y < z$ are a *Pythagorean triple* provided that

$$x^2 + y^2 = z^2.$$

Determine *all* Pythagorean triples where x, y, and z are consecutive natural numbers.

Solution 2
(Ungraded) This amounts to solving the equation

$$n^2 + (n + 1)^2 = (n + 2)^2,$$

which, after you expand it, is just a quadratic. The only positive solution is $n = 3$, which gives the triple $(3, 4, 5)$.

Exercise 3

- (**a**) Prove that

$$\sum_{k=0}^{n} a(k) = b(n)$$

for integers $n \geq 0$ if and only iff

$$b(n+1) - b(n) = a(n+1); \quad b(0) = a(0).$$

[Hint: Use induction to get from the difference to the sum.]

(b) Use the above technique to show that the sum of the first n positive integers is $n(n+1)/2$.

(c) Use the above technique to show that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

[Hint: You know what the right-hand side is from the previous part.]

Solution 3

(a) (5 points) If

$$\sum_{k=0}^{n} a(k) = b(n),$$

then obviously $b(n+1) - b(n) = a(n+1)$ and $b(0) = a(0)$.

On the other hand, suppose that

$$b(n+1) - b(n) = a(n+1); \quad b(0) = a(0).$$

We will prove the summation identity via induction. The base case, $n = 0$, is assumed: $b(0) = a(0) = \sum_{k=0}^{0} a(k)$. For the inductive step, suppose that

$$\sum_{k=0}^{n} a(k) = b(n)$$

for some $n \geq 0$. Then,

$$\begin{aligned}\sum_{k=0}^{n+1} a(k) &= b(n) + a(n+1) \\ &= b(n+1).\end{aligned}$$

Therefore the equation $b(n) = \sum_{k=0}^{n} a(k)$ holds for *all* integers $n \geq 0$.

(b) (3 points) Just check that $\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = n+1$ and that $\frac{0(0+1)}{2} = 0$.

(c) (3 points) Check that $b(n) = \left(\frac{n(n+1)}{2}\right)^2$ satisfies $b(n+1) - b(n) = (n+1)^3$ and $b(1) = 1$ (or $b(0) = 0$).

Exercise 4 The *Fibonacci numbers* F_n are defined by the following recurrence:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n. \end{aligned}$$

(a) Compute F_n for $0 \leq n \leq 10$.

(b) Compute $\sum_{k=0}^{n} F_k$ from $n = 0$ to $n = 10$. Conjecture and prove a pattern here. [Hint: Exercise 3 might be helpful.]

Solution 4

(a) (2 points) 0, 1, 1, 2, 3, 5, 8, 13, 21, 34

(b) (5 points) Note that $F_{0+2} - 1 = 0 = \sum_{k=0}^{0} F_k$, and that

$$(F_{n+3} - 1) - (F_{n+2} - 1) = F_{n+1}.$$

Therefore, by Exercise 3, $\sum_{k=0}^{n} F_k = F_{n+2} - 1$.

Exercise 5

(a) Prove that $\gcd(a, b) = \gcd(a - b, b)$.

(b) Evaluate $\gcd(F_n, F_{n+1})$ for $n \geq 1$, where F_n is the nth Fibonacci number.

(c) Evaluate $\gcd(F_n, F_{n+2})$ for $n \geq 1$.

Solution 5

(a) (5 points) If d divides a and b, then d divides $a-b$ and b. Conversely, if d divides $a-b$ and b, then d divides b and $a=(a-b)+b$. In other words, the common divisors of (a,b) are the same as the common divisors of $(a-b,b)$; in particular, $\gcd(a,b)=\gcd(a-b,b)$.

(b) (5 points) Note that $\gcd(F_n,F_{n+1})=\gcd(F_n,F_{n+1}-F_n)=\gcd(F_n,F_{n-1})$. In other words, $\gcd(F_n,F_{n+1})$ is constant for all $n\geq 1$. Since $\gcd(F_1,F_2)=\gcd(1,2)=1$, we see that $\gcd(F_n,F_{n+1})=1$ for all $n\geq 1$.

(c) (2 points) Note that $\gcd(F_n,F_{n+2})=\gcd(F_n,F_{n+2}-F_n)=\gcd(F_n,F_{n+1})=1$, by the previous part.

Exercise 6 This exercise involves programming. Don't panic! I'll explain how this works in class on Thursday.

Let $a(n)=2^n+1$.

(a) Write a Python program to compute $a(n)$ for arbitrary n.

(b) Using your program, determine the remainder of $a(n)$ divided by 10 for for $1\leq n\leq 16$. (Recall that this is the last digit of $a(n)$ in base 10.) Guess a pattern.

(c) Do the same for the remainder of $a(n)$ divided by 7.

Exercise 7 What is your favorite integer? Enter it into http://www.numbergossip.com/ and give some of your favorite properties.

Solution 7

(Ungraded) My favorite integer is 6.

This is my favorite property of 6: In binary, $6=(110)_2$. There are an even number of 1's in the expansion. Also, the proper divisors of 6 are 1, 2, and 3. If you add these up, you get 6: $1+2+3=6$.

Amazingly, 6 *is the only even number that does both of these things at once.*