Number Theory Final Exam

RDB

July 7, 2022

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

— Carl Friedrich Gauss

INSTRUCTIONS   No outside materials (notes, textbook, internet) or resources (calculators). Write your name on the big line below. The exam is scored out of 180 points.

Good luck,

Name: _____!

Problem 1   Write 64 in base 2, base 3, and base 5. (5 points each)

Problem 2 Using the Euclidean algorithm, find the greatest common divisor of 138 and 50. (5 points)

Problem 3 Using the Euclidean algorithm, find integers x and y such that

$$5x + 93y = 1,$$

if any such x and y exist. (10 points)

Problem 4

(a) Prove that $\gcd(a, b) = \gcd(a - b, b)$ for all integers a and b. (10 points)

(b) Prove that $\gcd(F_{n+1}, F_n) = 1$ for $n \geq 1$, where F_n is the nth Fibonacci number. (10 points)

Problem 5 Give the general solution to the congruence equation

$$5x \equiv 1 \pmod{11},$$

if any solutions exist. (10 points)

Problem 6 How many solutions does $100x \equiv 7 \pmod{200}$ have in the set $\{0, 1, 2, \ldots, 199\}$?
(5 points)

Problem 7 Find the smallest positive integer x such that x is divisible by 10 and $x + 1$ is divisible by 9. (10 points)

Problem 8 Find the smallest positive solution x to the following system of equations, if any solutions exist:

$$x \equiv -2 \pmod{5}$$
$$x \equiv -1 \pmod{6}.$$

(10 points)

Problem 9 How many positive integers less than or equal to 27000 are relatively prime to 27000? (10 points)

Problem 10

(**a**) How many divisors does 2022 have? (5 points)

(**b**) What is the sum of those divisors? (5 points)

[Hint: $2022 = 2 \cdot 3 \cdot 337$ and 337 is prime.]

Problem 11 How many primitive roots are there mod 7? (10 points)

Problem 12 What is the remainder of 5^{105} when divided by 12? (10 points)

Problem 13 Prove that $5^k \equiv 5 \pmod 7$ when $k \equiv 1 \pmod 6$. (10 points)

Problem 14

(**a**) State the law of quadratic reciprocity. (5 points)

(**b**) Is 5 a quadratic residue mod 7919? [Hint: 7919 is prime.] (5 points)

(**c**) Is 3 a quadratic residue mod 7919? [Hint: $7919 \equiv 3 \pmod 4$.] (5 points)

Problem 15

(**a**) How many quadratic residues are there mod 31? (10 points)

(**b**) Is 30 one of those quadratic residues? (10 points)

Problem 16 Is $x^2 + 3x + 1$ divisible by 37 for any integer x? (10 points)

BONUS QUESTION   The exam is scored out of 180 points, all available in the previous 16 questions. The following bonus question is worth an additional 12 points.

Problem 17   Let $S(p) = \sum_{k=1}^{p-1} \left(\frac{k}{p} \right)$ for a prime p.

(a) What is $S(3)$? (2 points)

(b) What is $S(5)$? (2 points)

(c) What is $S(7)$? (3 points)

(d) What is $S(p)$ for any prime p? You must prove your answer. (5 points)