Number Theory Midterm II

RDB

June 14, 2023

If you know the enemy and know yourself, your victory will not stand in doubt.
— Sun-Tzu (the other one)

INSTRUCTIONS No outside materials (notes, textbook, internet) or resources (calculators). Write your name on the big line below.

Good luck,

Name: ____!

Problem 1 Using the Chinese Remainder Theorem, find the smallest positive solution x to the following system of equations, if any exists:

$$
\begin{aligned}
x &= 1 \pmod 3 \\
x &= 6 \pmod 7.
\end{aligned}
$$

(10 points)

Problem 2 Find the smallest positive integer x such that x is divisible by 3, $x - 1$ is divisible by 5, and $x + 1$ is divisible by 7. (10 points)

Problem 3 Using the Chinese Remainder Theorem, find the smallest positive solution x to the following system of equations, if any exists:

$$\begin{aligned} 2x &= 3 \pmod 4 \\ 3x &= 1 \pmod{11}. \end{aligned}$$

(10 points)

Problem 4

(**a**) How many divisors does 980 have? (5 points)

(**b**) What is the sum of those divisors? (5 points)

Problem 5 Prove that the divisor function $d(n)$ is multiplicative. (10 points)

Problem 6 Let f be a multiplicative function. Prove that $f(n) = 1$ for all positive integers n iff $f(p^k) = 1$ for all primes p and nonnegative integers k. (10 points)

Problem 7 Let $\sigma(n)$ be the sum of divisors of n. Prove that

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}$$

for every prime power p^k. [Hint: $\sum_{j=0}^{k} x^j = \frac{x^{k+1}-1}{x-1}$ if $x \neq 1$.] (10 points)

Problem 8 What are the last two digits of 3^{44}? [Hint: The last two digits are the remainder when dividing by 100.] (10 points)

Problem 9 What is the remainder of $13^{(7^{42})}$ when divided by 15? (That's 13 to the 7^{42}, *not* $13^{7 \cdot 42}$.)

Problem 10 Prove that ab divides x if a and b divide x and $\gcd(a, b) = 1$. (10 points)

BONUS PROBLEM The exam is scored out of 100 points, all contained in the previous ten questions. The following is a bonus question worth a potentially infinite number of points.

Problem 11 A positive integer n is *perfect* provided that the sum of its proper divisors equals n. In other words, if $\sigma(n) = 2n$.

(**a**) (1 point each) Write as many perfect numbers as you know.

(**b**) (5 points) Prove that $\sum_{d|n} \frac{1}{d} = 2$ if n is perfect.