Number Theory Quiz IV

RDB

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Problem 1 What is the smallest positive integer x such that x is divisible by 4 and $x + 1$ is divisible by 7?

Solution 1
This is equivalent to the system

$$\begin{aligned} x &= 0 \pmod 4 \\ x &= -1 \pmod 7. \end{aligned}$$

Particular solutions to the individual equations are $x_1 = 0$ and $x_2 = 6$, respectively. Therefore, a particular simultaneous solution is

$$x_0 = 0 \cdot 7 \cdot 7^{-1} + 6 \cdot 4 \cdot 4^{-1},$$

where 7^{-1} is the inverse of 7 mod 4, and 4^{-1} is the inverse of 4 mod 7. The latter is 2, so

$$x_0 = 6 \cdot 4 \cdot 2 = 48.$$

By the CRT, every other solution is of the form

$$x = 48 + t4 \cdot 7 = 48 + 28t.$$

So the smallest *positive* solution is $48 - 28 = 20$.

Problem 2 Fix distinct primes p and q. Prove:

1. There exists some nonnegative x such that x is divisible by p and $x + 2$ is divisible by q.
2. That you can take $0 \leq x < pq^2$.

Solution 2

1. By the Chinese Remainder Theorem, the system

$$x = 0 \pmod p$$
$$x = -2 \pmod q$$

   has infinitely many solutions, all of the form $x_0 + pqt$ for some integer t. Taking t to be large enough gives $x_0 \geq 0$.

2. In fact, since $x_1 = 0$ and $x_2 = q - 2$ is a particular solution to the above equations, we can take

$$x_0 = x_2 p p^{-1},$$

   where p^{-1} is the inverse of p mod q. Whatever this is, it is less than q, so

$$0 \leq x_0 = (q-2)pp^{-1} < qpq = pq^2.$$