Number Theory Homework I

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This homework is meant to be a warmup for things covered in 300 along with a peek at some number theory. Direct proofs, inequalities, induction, the contrapositive, and perhaps most importantly, *style*.

Homework exists to give you practice. Try to solve the questions on your own before anything else; you will get the most out of it this way. (Plus, you won't have any outside resources on quizzes and exams!) Once you've given it a good try, feel free to collaborate with your classmates or ask me about it.

EXPECTATIONS Write your proofs with *good style*. What is good style? Concise yet enlightening. Simple yet pretty. Practical yet fun. Basically, go read the first chapter of Knuth's masterpiece and buy yourself a copy of Strunk and White's *The Elements of Style*. Your efforts will be repayed tenfold.

For example, suppose you were proving "the sum of even integers is even."

Proof [Terrible proof]

$$\begin{aligned}
&\exists n, k\ [2n + 2k] \\
&\quad \implies 2n + 2k = 2(n + k) \\
&\exists m\ [n + k = m] \\
&\therefore 2n + 2k = 2m
\end{aligned}$$

■

Proof [Excellent proof] If x and y are even integers, then there exist integers n and k such that

$$x + y = 2n + 2k = 2(n + k).$$

Since $n + k$ is an integer, $x + y$ is even. ■

Both proofs are (essentially) logically correct, yet one makes me dizzy. Bring your readers the joy of discovery, not the pain of parsing notation.

Exercises

In general, assume that variables like n, m, and k are integers.

Exercise 1

(**a**) Prove that n^2 is even if and only if n is even.

(**b**) If $\sqrt{2} = a/b$ for coprime integers a and b, then $2b^2 = a^2$. Use the previous part to derive a contradiction about a and b, and conclude that $\sqrt{2}$ is irrational.

Solution 1

(**a**) (3 points) If $n = 2k$, then $n^2 = 4k^2 = 2(2k^2)$. On the other hand, if $n = 2k + 1$, then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

(**b**) (3 points) Since $2b^2 = a^2$, we see that a^2 is even. The previous part implies that a is even, say $a = 2k$. Then $2b^2 = 4k^2$, and dividing by 2 yields $b^2 = 2k^2$. But then b is even, and that contradicts that a and b are coprime.

Exercise 2 Three natural numbers $x < y < z$ are a *Pythagorean triple* provided that

$$x^2 + y^2 = z^2.$$

(**a**) Determine *all* Pythagorean triples where x, y, and z are consecutive natural numbers.

(**b**) Given a natural number a, let's say that n is an *a-Pythagorean integer* if

$$n^2 + (n + a)^2 = (n + 2a)^2.$$

For a fixed a, how many a-Pythagorean integers are there?

Solution 2

(**a**) (3 points) This amounts to solving the equation

$$n^2 + (n + 1)^2 = (n + 2)^2,$$

which, after you expand it, is just a quadratic. The only positive solution is $n = 3$, which gives the triple $(3, 4, 5)$.

(**b**) (3 points) The equation

$$n^2 + (n+a)^2 = (n+2a)^2$$

is a quadratic in n and has a single positive solution for $a > 0$, namely $n = 3a$. So there is only *one* a-Pythagorean integer for every a.

Exercise 3

(**a**) Prove that

$$\sum_{k=0}^{n} a(k) = b(n)$$

is equivalent to

$$b(n+1) - b(n) = a(n+1); \quad b(0) = a(0).$$

[Hint: Use induction to get from the difference to the sum.]

(**b**) Use the above technique to show that the sum of the first n positive integers is $n(n+1)/2$.

(**c**) Use the above technique to show that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

[Hint: You know what the right-hand side is from the previous part.]

Solution 3

(**a**) (4 points) If

$$\sum_{k=0}^{n} a(k) = b(n),$$

then obviously $b(n+1) - b(n) = a(n+1)$ and $b(0) = a(0)$.

On the other hand, suppose that

$$b(n+1) - b(n) = a(n+1); \quad b(0) = a(0).$$

We will prove the summation identity via induction. The base case, $n = 0$, is assumed: $b(0) = a(0) = \sum_{k=0}^{0} a(k)$. For the inductive step, suppose that

$$\sum_{k=0}^{n} a(k) = b(n)$$

for some $n \geq 0$. Then,

$$\begin{aligned}\sum_{k=0}^{n+1} a(k) &= b(n) + a(n+1) \\ &= b(n+1).\end{aligned}$$

Therefore the equation $b(n) = \sum_{k=0}^{n} a(k)$ holds for *all* integers $n \geq 0$.

(b) (2 points) Just check that $\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = n + 1$ and that $\frac{0(0+1)}{2} = 0$.

(c) (2 points) Check that $\left(\frac{n(n+1)}{2}\right)^2$ satisfies the right equations.

Exercise 4 The *Fibonacci numbers* F_n are defined by the following recurrence:

$$\begin{aligned}F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n.\end{aligned}$$

(a) Compute the first ten Fibonacci numbers.

(b) Prove that

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1.$$

[Hint: Use the earlier exercise about sums. Or induction.]

(c) The *square* Fibonacci numbers $S_n = F_n^2$ *also* satisfy a nice recurrence:

$$S_{n+3} = 2S_{n+2} + 2S_{n+1} - S_n.$$

Check that this is true up to $n = 6$.

Solution 4

(**a**) (2 points) 0, 1, 1, 2, 3, 5, 8, 13, 21, 34

(**b**) (2 points) Note that $F_{0+2} - 1 = 0 = \sum_{k=0}^{0} F_k$, and that

$$(F_{n+3} - 1) - (F_{n+2} - 1) = F_{n+1}.$$

(**c**) (2 points) The first ten squares are: 0, 1, 1, 4, 9, 25, 64, 169, 441, 1156. Using the recurrence

$$S_{n+3} = 2S_{n+2} + 2S_{n+1} - S_n$$

along with the initial values $S_0 = 0$, $S_1 = 1$, $S_2 = 1$, we get the same list:

$$\begin{aligned} 2(1) + 2(1) - 0 &= 4 \\ 2(4) + 2(1) - 1 &= 9 \\ 2(9) + 2(4) - 1 &= 25 \\ 2(25) + 2(9) - 4 &= 64, \end{aligned}$$

and so on.

Exercise 5 Prove that $n! > 2^n$ for sufficiently large n. [Hint: This means that there exists some N such that $n! > 2^n$ for $n \geq N$. You have to find that N and then prove it. Induction is the way to go here.]

Solution 5
(4 points)

Suppose that $n! > 2^n$ for some $n \geq N$. [We haven't yet determined N, but that's OK! We'll come back to it.] Then,

$$(n+1)! = (n+1) \cdot n! > (n+1)2^n.$$

So, if $n + 1 > 2$, or $n > 1$, then $(n+1)! > 2^{n+1}$. In other words, the inductive argument will work as long as we start at $N = 2$.

But we need a good base case, and $N = 2$ doesn't work. Note that $4! = 24 > 16 = 2^4$, so $N = 4$ will do the job.

Exercise 6 This exercise involves programming. When you write a program, save it as a ".py" file and submit it with your homework. (I'll explain how to do this in class on Thursday if this doesn't make sense to you.)

Let $a(n) = 2^n + 1$.

(a) Write a Python program to compute $[a(1), a(2), a(3), \ldots, a(n)]$ for arbitrary n. [Hint: Lookup "python list comprehension."]

(b) Using your program, determine which values of $a(n)$ are prime for $1 \leq n \leq 16$. Guess a pattern.

(c) Prove that $a(n)$ is *not* prime if n is odd. [Hint: Prove that 3 divides $a(n)$ if n is odd, probably by induction.] Does this prove your pattern? Explain.

Solution 6

(a) (2 points) Simple python program: `[2**k + 1 for k in range(1, 17)]`. There are infinitely many ways you could have structured this.

(b) (2 points) In the given range, the n for which $a(n)$ is prime are 1, 2, 4, 8, and 16. It *seems* like $a(n)$ might be prime when n is a power of 2.

(c) (3 points) One way is induction. Our statement is "3 divides $2^{2n+1} + 1$ for all non-negative integers n." The base case, $n = 0$, is obvious, since $2^{2 \cdot 0+1} + 1 = 3$. Suppose that $2^{2n+1} + 1 = 3k$ for some integer k. Then, we have

$$2^{2(n+1)+1} + 1 = 4 \cdot 2^{2n+1} + 1.$$

Our assumption then gives

$$\begin{aligned} 4 \cdot 2^{2n+1} + 1 &= 4(3k - 1) + 1 \\ &= 12k - 4 + 1 \\ &= 3(3k - 1), \end{aligned}$$

so 3 divides $2^{2(n+1)+1} + 1$ as well.

Exercise 7 What is your favorite integer? Enter it into http://www.numbergossip.com/ and give some of your favorite properties.

Solution 7

(3 points) My favorite integer is 6. My favorite property listed on the site is something I don't know how to prove.

In binary, $6 = (110)_2$. There are an even number of 1's in the expansion.

Also, the proper divisors of 6 are 1, 2, and 3. If you add these up, you get 6:

$$1 + 2 + 3 = 6.$$

Supposedly, 6 *is the only even number that does both of these things at once.* This seems surprising, but maybe it's actually easy to prove. I don't know!