Number Theory Final Exam

RDB

August 18, 2021

INSTRUCTIONS No outside materials (notes, textbook, internet) or resources (calculators). Leave your webcam on until you submit and I confirm that I have your exam.

Good luck!

Problem 1 How many positive integers less than 21000 are relatively prime to 21000?

Problem 2 Give the general solution to the congruence equation

$$7x \equiv 1 \pmod{11},$$

if any solutions exist.

Problem 3 Prove that $3^{(p-1)/2} \equiv \pm 1 \pmod{p}$ if $p > 3$ is prime.

Problem 4 Using the Euclidean algorithm, find the greatest common divisor of 138 and 52.

Problem 5 Using the Euclidean algorithm, find integers x and y such that

$$17x + 93y = 1,$$

if any such x and y exist.

Problem 6

(**a**) Prove that $\gcd(a, b) = \gcd(a - b, b)$ for all integers a and b.

(**b**) Prove that $\gcd(F_{n+1}, F_n) = 1$ for $n \geq 1$, where F_n is the nth Fibonacci number.

Problem 7 How many primitive roots are there mod 127? [Hint: 127 is prime.]

Problem 8 Let a be an integer relatively prime to the positive integer n. Prove that $|a|_n$, the multiplicative order of a mod n, divides $\phi(n)$.

Problem 9

(**a**) Why is $f(n) = \sum_{d|n} \mu(d)$ multiplicative?

(**b**) Prove that $f(1) = 1$ and $f(n) = 0$ if $n > 1$. [Hint: Prove that $f(p^k) = 0$ if $k \geq 1$, then apply multiplicativity.]

Problem 10

(**a**) State Euler's theorem.

(**b**) What is the remainder of 3^{702} when divided by 11?

Problem 11 How many mutually incongruent solutions does $100x \equiv 7 \pmod{200}$ have?

Problem 12 Prove that $2^n \equiv -1 \pmod{3}$ if n is odd.

Problem 13 Using Euler's criterion, prove that -1 is not a quadratic residue mod 11.

Problem 14 Prove or disprove the following statement: For every integer m, there exists exactly one integer $0 \leq x < m$ such that $x^2 \equiv 0 \pmod{m}$.

Problem 15 Using the Chinese Remainder Theorem (with the formula for a particular solution!), find the smallest positive solution x to the following system of equations, if any solutions exist:

$$\begin{aligned} x &= 2 \pmod{5} \\ x &= 1 \pmod{6}. \end{aligned}$$

Problem 16 Let f be a multiplicative function. Prove that $f(n) = 1$ for all positive integers n iff $f(p^k) = 1$ for all primes p and nonnegative integers k.

Problem 17 Suppose that

$$ax + by = 1,$$

where a, b, x, and y are all integers. Prove that $\gcd(a, b) = 1$.

Problem 18 Write 64 in base 2, base 3, and base 5.

Problem 19

(**a**) State Euclid's lemma.

(**b**) Prove, using Euclid's lemma, that $ax \equiv bx \pmod{p}$ implies $a \equiv b \pmod{p}$ if p does not divide x.

Problem 20 Using the Chinese Remainder Theorem (with the formula for a particular solution!), find the smallest positive integer x such that x is divisible by 10 and $x + 1$ is divisible by 9.