Number Theory Midterm II

RDB

August 9, 2021

INSTRUCTIONS No outside materials (notes, textbook, internet) or resources (calculators). Leave your webcam on until you submit and I confirm that I have your exam.

Good luck!

Problem 1 Using the Chinese Remainder Theorem, find the smallest positive solution x to the following system of equations, if any exists:

$$\begin{aligned} x &= 1 \pmod 3 \\ x &= 6 \pmod 7. \end{aligned}$$

(10 points)

Problem 2 Find the smallest positive integer x such that x is divisible by 3, $x-1$ is divisible by 5, and $x+1$ is divisible by 7. (10 points)

Problem 3 Using the Chinese Remainder Theorem, find the smallest positive solution x to the following system of equations, if any exists:

$$\begin{aligned} 2x &= 3 \pmod 4 \\ 3x &= 1 \pmod{11}. \end{aligned}$$

(10 points)

Problem 4

(**a**) How many divisors does 980 have? (5 points)

(**b**) What is the sum of those divisors? (5 points)

Problem 5 Find the unique integer $r \in \{0, 1, 2, \ldots, 29\}$ such that

$$7^{242} \equiv r \pmod{30}.$$

(10 points)

Problem 6

(**a**) Write out the correspondance $x \mapsto (x \bmod a, x \bmod b)$ for $a = 2$ and $b = 3$ and $0 \leq x < 6$. (5 points)

(**b**) Which pairs represent integers relatively prime to 6? (5 points)

Problem 7 Let f be a multiplicative function. Prove that $f(n) = 1$ for all positive integers n iff $f(p^k) = 1$ for all primes p and nonnegative integers k. (10 points)

Problem 8 Let $\sigma(n)$ be the sum of divisors of n. Prove that

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}$$

for every prime power p^k. [Hint: $\sum_{j=0}^{k} x^j = \frac{x^{k+1}-1}{x-1}$ if $x \neq 1$.] (10 points)

Problem 9 Let $d(n)$ be the number of divisors of n. Prove that $d(p^k) = k + 1$ for every prime power p^k. (10 points)

Problem 10

(**a**) Define the Möbius function $\mu(n)$. (5 points)

(**b**) Using your definition, compute

$$\mu(1) + \mu(2) + \cdots + \mu(10).$$

(5 points)

Problem 11 Prove that ab divides x if a and b divide x and $\gcd(a, b) = 1$. (10 points)

BONUS PROBLEM The exam is scored out of 110 points, all contained in the previous eleven questions. The following is a bonus question worth 10 points.

Problem 12 Let

$$f(n) = \sum_{t|n} d(t),$$

where $d(n)$ is the number of divisors of n.

(**a**) Why is f multiplicative? (3 points)

(**b**) Prove that $f(p^k) = \frac{(k+1)(k+2)}{2}$ for any prime power p^k. (4 points)

(**c**) Evaluate $f(100)$. (3 points)