Number Theory Midterm I

RDB

August 9, 2021

I feel fine today modulo a slight headache.
— *The Hacker's Dictionary*

INSTRUCTIONS No outside materials (notes, textbook, internet) or resources (calculators). Leave your webcam on until you submit and I confirm that I have your exam.

Good luck!

Problem 1 Use the Euclidean algorithm to compute the greatest common divisor of 153 and 64. (5 points)

Problem 2

(**a**) State Euclid's lemma. (3 points)

(**b**) Show that, if a prime p divides a^2 for some integer a, then p divides a. (5 points)

Solution 2

(**a**) If $p|ab$ for a prime p and integers a and b, then $p|ab$.

(**b**) If $p|a^2 = a \cdot a$, then $p|a$ or $p|a$. So $p|a$, clearly.

Problem 3 Find the general solution to

$$42x + 35y = 2,$$

if any solutions exist. (10 points)

Solution 3
The gcd of 42 and 35 is 7, which does not divide 2, so there are no solutions.

Problem 4 List three solutions to the congruence $2x \equiv 4 \pmod{101}$. Your solutions may be congruent mod 101. (10 points)

Solution 4
An obvious solution is $x = 2$, but there are no others in $\{0, 1, 2, \ldots, 100\}$, because $\gcd(2, 101) = 1$. We can still get other solutions by adding multiples of 101. In particular, $2 + 101 = 103$ and $2 + 202 = 204$ are also solutions.

Problem 5 Find the general solution to

$$38x + 15y = 1,$$

if any solutions exist. (10 points)

Solution 5
The gcd of 38 and 15 is 1, so solutions *do* exist. The Euclidean algorithm will give $x_0 = 2$ and $y_0 = -5$, so the general solution is

$$x = 2 + 15t; \qquad y = -5 - 38t$$

for an integer t.

Problem 6 Does the congruence $25x \equiv 1 \pmod{1000}$ have solutions? If so, how many solutions in $\{0, 1, 2, 3, \ldots, 999\}$ are there? (10 points)

Solution 6
No, because $\gcd(25, 1000) = 25$, which does not divide 1.

Problem 7 Show that $\gcd(a, b) = \gcd(a - b, b)$ for any integers a and b. (10 points)

Solution 7
If d is a common divisor of a and b, say $a = dk$ and $b = dj$, then $a - b = d(k - j)$, so d is a common divisor of $a - b$ and b. Conversely, if d is a common divisor of $a - b$ and b, then it also divides $a = (a - b) + b$ by the same argument. Therefore the common divisors of (a, b) are the same as the common divisors of $(a - b, b)$, and in particular the *greatest* one of them is the same.

Problem 8 Show that $2^n \equiv -1 \pmod{3}$ for n odd. (10 points)

Solution 8
Note that $2 \equiv -1 \pmod 3$, so $2^n \equiv (-1)^n| \pmod 3$. If n is odd $(-1)^n = -1$, so $2^n \equiv -1 \pmod 3$ for odd n.

Nearly everyone used induction, which was fine, but gross. Here's the best induction you could do, I think: If $2^n \equiv -1 \pmod 3$, then

$$3^{n+2} \equiv -4 \pmod 3 \equiv -1 \pmod 3.$$

Problem 9 Fix integers a and b. Suppose that $ax + by = p$ for some integers x and y and a prime p. Prove that $\gcd(a, b)$ is either 1 or p. (10 points)

Solution 9
By a theorem in class (or writing $a = \gcd(a, b)k$ and $b = \gcd(a, b)j$), we see that $\gcd(a, b)$ divides p, so it is either 1 or p since p is prime.

Problem 10 Write 76 in base 3. (5 points)

Solution 10
The largest power to begin with is $3^3 = 27$, and we can fit 2 of them in:

$$76 - 2 \cdot 3^3 = 22.$$

Repeating:

$$22 - 2 \cdot 3^2 = 4$$

Again:

$$4 - 1 \cdot 3 = 1$$

Finally:

$$1 = 1 \cdot 3^0.$$

So

$$\begin{aligned} 76 &= 2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^1 + 1 \cdot 3^0 \\ &= (2211)_3. \end{aligned}$$

Problem 11 Translate the following numbers from binary into base 10:

(a) $(1)_2$ (1 point)

(b) $(11)_2$ (1 point)

(**c**) $(111)_2$ (1 point)

(**d**) $(1111)_2$ (1 point)

(**e**) Prove, by induction, that

$$(\underbrace{11\cdots1}_{n})_2 = 2^n - 1$$

for all positive integers n. (5 points)

Problem 12 Write $(1024)_5$ in base 10. (5 points)

Solution 12

$$\begin{aligned}(1024)_5 &= 1\cdot5^3 + 0\cdot5^2 + 2\cdot5^1 + 4\cdot5^0\\ &= 125 + 10 + 4\\ &= 139.\end{aligned}$$

BONUS PROBLEM The exam is graded out of 102 points, all accounted for in the previous problems. The following problem is worth an additional 10 points.

Problem 13

(**a**) Find the general solution to the diophantine equation

$$10x + 11y = 200.$$

(2 points)

(**b**) Show that there exists only a single solution with x and y both positive. (5 points)

(**c**) Find that positive solution. (3 points)